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Exact Solutions to a Coupled Nonlinear Equation

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A coupled nonlinear partial differential equation is studied which represents a model for wave propagation in a one-dimensional nonlinear lattice in the absence of one of the variables. The coupled equation is solved exactly by applying the criteria of the Weierstrass elliptic function.

Recently considerable attention has been focused on the study of coupled nonlinear partial differential equations (Guha-Roy *et al.*, 1986; Guha-Roy, 1987a,b; Krishnan, 1982, 1986) that can be solved exactly. Here we study the following coupled nonlinear equation (Guha-Roy, 1987b):

$$\Phi_t + \alpha\Psi^2\Psi_x + \beta\Phi_x + \lambda\Phi\Phi_x + \gamma\Phi_{xxx} = 0 \tag{1a}$$

$$\Psi_t + \delta(\Phi\Psi)_x + \varepsilon\Psi\Psi_x = 0 \tag{1b}$$

where the subscripts refer to partial differentiations with respect to the indicated variables, and α, β, λ, γ, δ, and ε are arbitrary parameters. It is interesting to point out that for $\Psi = 0$ equation (1) represents a model for wave propagation in a one-dimensional nonlinear lattice. Furthermore, as is outlined in Wadati (1975), for $\Psi = 0$, equation (1) shares properties with the KdV equation ane the modified KdV equation, under certain conditions.

Our main concern in the present paper is to seek exact solutions of (1) by applying the criteria of the Weierstrass elliptic function. The solitary wave solution will be obtained as a simple limit of a stationary periodic solution.

In a recent paper (Guha-Roy *et al.*, 1986) we have shown, by introducing an analogue of the stream function, that if one of the solutions of some coupled nonlinear equations is of the traveling wave type, then the other

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must also exhibit the same form. Keeping this in mind, we choose a new variable s $(=x-ct)$ such that

$$\Phi\equiv\Phi(s),\qquad \Psi\equiv\Psi(s) \tag{2}$$

where c is the constant speed of propagation.

Using (2), we integrate (1) once to obtain

$$-c\Phi+\frac{\alpha}{3}\Psi^3+\frac{\beta}{3}\Phi^3+\frac{\lambda}{2}\Phi^2+\gamma\Phi''=k_1 \tag{3a}$$

$$-c\Psi+\delta\Phi\Psi+\frac{\varepsilon}{2}\Psi^2=k_2 \tag{3b}$$

In equation (3a), the primes denote differentiation with respect to s; k_1 and k_2 are constants of integration. It is to be noted here that Φ would be regular everywhere provided k_2 vanishes. As a result, equation (3b) yields

$$\Psi=2\left(\frac{c}{\varepsilon}-\frac{\delta}{\varepsilon}\Phi\right) \tag{4}$$

Inserting (4), one can eliminate Ψ from (3a) to get

$$\gamma\Phi''-\frac{1}{3\varepsilon^3}(8\alpha\delta^3-\beta\varepsilon^3)\Phi^3+\frac{1}{2\varepsilon^3}(16\alpha c\delta^2+\lambda\varepsilon^3)\Phi^2$$
$$-\frac{1}{\varepsilon^3}(8\alpha c^2\delta+c\varepsilon^3)\Phi+\frac{8\alpha c^3}{3\varepsilon^3}=k_1 \tag{5}$$

Now, from the vanishing boundary conditions

$$\Phi,\Phi',\Phi''\to 0\qquad \text{as}\quad |s|\to\infty$$

k_1 may be determined as

$$k_1=8\alpha c^3/3\varepsilon^3$$

Thus, equation (5) may be expressed as

$$\Phi''=\eta_1\Phi-\tfrac{1}{2}\eta_2\Phi^2+\tfrac{1}{3}\eta_3\Phi^3 \tag{6}$$

where

$$\eta_1=(8\alpha c^2\delta+c\varepsilon^3)/\gamma\varepsilon^3$$
$$\eta^2=(16\alpha c\delta^2+\lambda\varepsilon^3)/\gamma\varepsilon^3$$
$$\eta_3=(8\alpha\delta^3-\beta\varepsilon^3)/\gamma\varepsilon^3$$

From equation (6), it is obvious that the solutions depend effectively on the values of η_1, η_2, and η_3. In the following we adopt the methodology of Kano and Nakayama (1981) to work out the solutions of (6). As such, we seek (Krishnan, 1982, 1986) a solution of Φ in the form

$$\Phi(s) = a(p(s)/q(s)) \tag{7}$$

where $p(s)$ is the Weierstrass elliptic function, $q(s) = 1 + bp(s)$, and a and b are arbitrary parameters. Consequently, $p(s)$ satisfies the condition

$$(p')^2 = 4p^3 - 2g_2 p - g_3 \tag{8}$$

In (8), g_2 and g_3 are both real constants such that $8g_2^3 > 27g_3^2$.

We next substitute equation (7) into (6) and then equate the coefficients of the powers of p on both sides. This yields the following relations:

$$\eta_1 b^2 - \tfrac{1}{2}\eta_2 ab + \tfrac{1}{3}\eta_3 a^2 = -2b \tag{9a}$$

$$2\eta_1 b - \tfrac{1}{2}\eta_2 a = 6 \tag{9b}$$

$$\eta_1 = 3bg_2 \tag{9c}$$

$$0 = 2bg_3 - g_2 \tag{9d}$$

Now, from equations (9a) and (9b) we can easily determine a and b as

$$a = \frac{[12\eta_2 + 48(\eta_2^2 - 5\eta_1\eta_3)^{1/2}]}{3\eta_2^2 - 16\eta_1\eta_3}, \qquad b = \frac{\eta_2 a + 12}{4\eta_1} \tag{10}$$

Moreover, g_2 and g_3 can be evaluated from (9c) and (9d). We find

$$g_2 = \eta_1/3b, \qquad g_3 = g_2/2b \tag{11}$$

Therefore, we can write the exact periodic solution as

$$\Phi(s) = \frac{ap(s+\theta;\, g_2,\, g_3)}{1 + bp(s+\theta;\, g_2,\, g_3)} \tag{12}$$

where θ is a constant of integration of (8) and a, b, g_2, and g_3 are expressed, respectively, by (10) and (11). As such, the exact bounded periodic solution can be obtained as

$$\Phi(s) = a\frac{e_3 + (e_2 - e_3)\,\mathrm{sn}^2[(e_1 - e_3)^{1/2}s + \theta_0]}{1 + b\{e_3 + (e_2 - e_3)\,\mathrm{sn}^2[(e_1 - e_3)^{1/2}s + \theta_0]\}} \tag{13}$$

where e_1, e_2, and e_3 are real roots of $4y^3 - 2g_2 y - g_3 = 0$ such that $e_3 < e_2 < e_1$ and sn is the Jacobian elliptic sine function; θ_0 is an arbitrary real parameter.

Noting that the modulus of sn is given by $m = (e_2 - e_3)/(e_1 - e_3)$, one can easily go to the solitary wave limit. Since the solitary wave is a wave when the period is infinite, we have $m = 1$. Thus $e_1 = e_2$. As a result

$$\Phi(s) = a\frac{e_1 - (e_1 - e_3)\,\mathrm{sech}^2[(e_1 - e_3)^{1/2}s + \theta_0]}{1 + b\{e_1 - (e_1 - e_3)\,\mathrm{sech}^2[(e_1 - e_3)^{1/2}s + \theta_0]\}} \tag{14}$$

which represents the solitary wave solution of (1).

In summary, we have obtained the exact solution of the coupled nonlinear partial differential equations (1a) and (1b). We have also found that under certain conditions this solution gives rise to the solitary wave solution. The knowledge of such solutions may have crucial significance in understanding the relevant features of nonlinear systems.

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REFERENCES

Guha-Roy, C. (1987a). *Journal of the Physical Society of Japan* **56**, 1625.
Guha-Roy, C. (1987b). *Journal of Mathematical Physics,* in press.
Guha-Roy, C., Bagchi, B., and Sinha, D. K. (1986). *Journal of Mathematical Physics,* **27**, 2558.
Kano, K., and Nakayama, N. (1981). *Journal of the Physical Society of Japan,* **50**, 361.
Krishnan, E. V. (1982). *Journal of the Physical Society of Japan,* **51**, 2391.
Krishnzn, E. V. (1986). *Journal of the Physical Society of Japan,* **55**, 3753.
Wadati, M. (1975). *Journal of the Physical Society of Japan,* **38**, 673.